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A generalized thermoelasticity problem of multilayered conical shells

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Abstract

The formulation begins with the basic equations of thermoelasticity in curvilinear circular conical coordinates. A method based on a hybrid Laplace transformation and finite difference method is developed to obtain the two-dimensional axisymmetric quasi-static coupled thermoelastic problems of laminated circular conical shells. It was shown that the solutions are rapidly convergent. Solutions for the temperature, displacement and thermal stress distributions in both transient and steady state are obtained. The present method can obtain stable solutions at a specific time; thus it is a powerful and efficient method to solve the coupled transient thermoelastic problems of a circular multilayered conical shell.

Keywords: Thermoelasticity; Multilayers; Conical shell; Laplace transform; Finite difference

1. Introduction

Truncated circular multilayered conical shells are used in various engineering applications such as hoppers, vessel heads, components of missiles and spacecrafts, and other civil, mechanical and aerospace engineering structures. Since the laminated circular conical structures are widely used in contemporary industries, we must take care the thermoelasticity problems. The literature on the thermoelastic problems of laminated circular conical shells is scarce. This is mainly due to the inherent complexity of the basic equations in curvilinear circular conical coordinates, which is a system of nonlinear partial differential equations.

Exact methods of solution are available in the literature for axisymmetric bending of conical shells as reported in many textbooks, for examples, Timoshenko and Woinowsky-Krieger (1959) and Flügge (1960). The exact solutions are based on Bessel functions, which are presented in terms of Kelvin's or Thomson's functions. However Kelvin's functions oscillate from positive to negative values with amplitudes of oscillation that diverge exponentially. Jianpong and Harik (1990) presented an iterative finite difference method

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to determine the stresses and displacements of bending of axisymmetric conical shells. The method can be applied to short and long conical shells having simply supported, clamped, or free edges and can easily be extended to tapered conical shells and other types of axisymmetric shells.

Based on a thick shell theory, Lu et al. (1995) discussed the stress distribution of thick laminated conical tubes under general loading. The effect of transverse shear is taken into account by a first-order theory. Governing equations are solved by a semi-analytical method that is a combination of Fourier series expansion, finite difference scheme and Riccati transfer matrix method. The method can be applied to the analysis of any axisymmetric laminated tube or shell that may approximately be divided into a series of conical shell segments. The expressions of determining the stresses, strains and displacements of a truncated or complete thin conical shell with constant thickness and axisymmetric load distributed or concentrated along the meridian was presented by Tavares (1996). These expressions were obtained by construction of Green's function for the homogeneous differential equation based on the bending theory. It is shown that a complete cone with lateral load can be obtained as a particular case and in a second step a cone with load at the vertex. The solution for an axisymmetric load distributed along the meridian was obtained using superposition.

An asymptotic theory for thermoelastic analysis of doubly curved laminated shells is formulated by Wu et al. (1996), which based on the framework of three-dimensional elasticity. The essential feature of the theory is that an accurate elasticity solution can be determined hierarchically by solving the classical laminated shell theory equations in a consistent way without treating the layers individually. Based on the equations of three-dimensional elasticity, an asymptotic theory for the analysis of laminated circular conical shells was presented by Wu and Hung (1999). By means of proper asymptotic expansion, they obtained the recursive sets of governing equations for the bending of a laminated circular conical shell. The differential quadrature method is used to solve the governing equations. The numerical solutions asymptotically approach to the three-dimensional solution. Furthermore, based on the equations of three-dimensional elasticity in curvilinear circular conical coordinates, a refined asymptotic theory for the static analysis of laminated circular conical shells was presented by Wu et al. (2002). Taking into account of the effect of transverse shear deformations, they obtained the recursive sets of governing equations leading to the ones of first-order shear deformation theory. The differential quadrature method is used to determine the asymptotic solutions for various orders. The solutions applied to laminated shells revealed that both the differential quadrature method and the asymptotic solution are rapidly convergent.

Based on the governing equations of three-dimensional elasticity, an asymptotic theory was presented by Wu and Chiu (2001) for the thermoelastic buckling analysis of laminated composite conical shells subjected to a uniform temperature change. The perturbation method is used to determine the critical thermal loads. Performing a straightforward derivation, the asymptotic formulation leads to recursive sets of governing equations for various orders. The critical thermal loads of simply supported, cross-ply conical shells are studied to demonstrate the performance of the asymptotic theory. The buckling of an orthotropic composite truncated conical shell with continuously varying thickness subjected to a time dependent external pressure was discussed by Sofiyev (2003). Using the Galerkin method, the governing equations have been reduced to time dependent differential equation with variable coefficients. Finally, applying the Ritz method, the critical static and dynamic loads, the corresponding wave numbers and the dynamic factor have been found analytically. It was observed that the critical parameters and static critical load have appreciable effects on the critical parameters of the problem in the heating.

Jane and Lee (1999) considered the thermoelasticity of multilayered cylinders subjected to known temperatures at traction-free boundaries by using the Laplace transform and the finite difference method. The computational procedures can solve the generalized thermoelasticity problem for a multilayered composite cylinder with non-homogeneous materials.

However, there are fewer research works about coupled thermoelastic problems of the laminated circular conical shells, due to the difficulties in theoretical analysis and complexity of mathematics. The present study deals with the two-dimensional quasi-static coupled thermoelastic problems of laminated circular conical shells composed of different multilayered materials having axis symmetry. The laminated circular conical shell is characterized by trace free, and absence of body forces and internal heat sources. Derivatives are approximated by central differences resulting in an algebraic representation of the partial differential equation. By taking the Laplace transform with respect to time, the general solutions in the transform domain are first obtained. The final solutions in the real domain can be obtained by inverting the Laplace transform.

2. Formulation

Consider a laminated circular conical shell composed of multiple layers with different materials. A set of the orthogonal curvilinear coordinates (η, θ, ζ) is located on the middle surface as shown in Fig. 1, in which η is the meridional direction, θ is the circumferential direction, and ζ is the normal direction. Let R_1 and R_2 be the radii of the cone at the small and large edges, respectively, α be the semi-vertex angle of the cone and L be cone length along the generator. The inner and outer temperatures are assumed to be f_1 and f_2, respectively. Temperatures at both ends are assumed to be f_3 and f_4, respectively.

The transient heat conduction equation for the kth layer of the conical shell can be written as

$$\begin{aligned} & k_\eta \frac{\partial^2 \overline{\Theta}}{\partial \eta^2} + k_\eta \frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha} \frac{\partial \overline{\Theta}}{\partial \eta} + k_\zeta \frac{\partial^2 \overline{\Theta}}{\partial \zeta^2} + k_\zeta \frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha} \frac{\partial \overline{\Theta}}{\partial \zeta} \\ & \quad = \rho C_v \frac{\partial \overline{\Theta}}{\partial \tau} + \Theta_0 \beta_\eta \frac{\partial}{\partial \eta}\left(\frac{\partial U_\eta}{\partial \tau}\right) + \Theta_0 \beta_\theta \frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha} \frac{\partial U_\eta}{\partial \tau} + \Theta_0 \beta_\theta \frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha} \\ & \quad \times \frac{\partial U_\zeta}{\partial \tau} + \Theta_0 \beta_\zeta \frac{\partial}{\partial \zeta}\left(\frac{\partial U_\zeta}{\partial \tau}\right) \end{aligned} \tag{1}$$

in which $\overline{\Theta} = \Theta - \Theta_0$, and

$$\beta_\eta = (1/A)[E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})\alpha_\eta + E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})\alpha_\theta + E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta})\alpha_\zeta]$$

$$\beta_\theta = (1/A)[E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})\alpha_\eta + E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta})\alpha_\theta + E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})\alpha_\zeta]$$

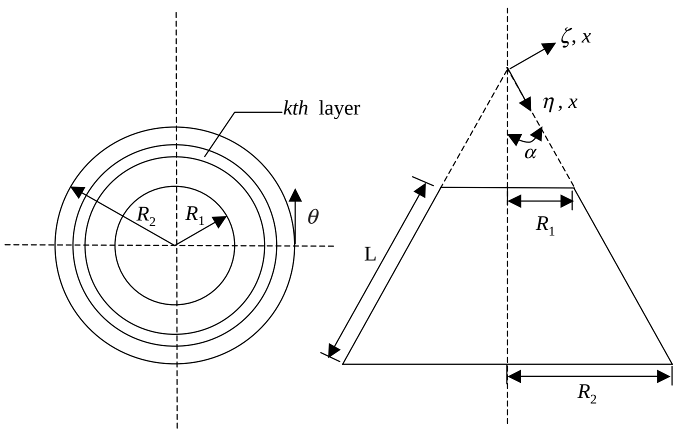

Fig. 1. Physical model and system coordinates for the circular conical shell.

$$\beta_\zeta = (1/A)[E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta})\alpha_\eta + E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})\alpha_\theta + E_\zeta(1 - v_{\eta\theta}v_{\theta\eta})\alpha_\zeta]$$

$$A = 1 - v_{\eta\theta}v_{\theta\eta} - v_{\theta\zeta}v_{\zeta\theta} - v_{\zeta\eta}v_{\eta\zeta} - 2v_{\theta\eta}v_{\zeta\theta}v_{\eta\zeta}$$

where Θ is the dimensional temperature; Θ_0 is the referential temperature; k_η and k_ζ are the thermal conductivities; τ is the dimensional time; ρ is the density; C_v is the specific heat; $v_{\eta\zeta}$, $v_{\eta\theta}$ and $v_{\theta\zeta}$ are Poisson's ratio; α_η, α_θ and α_ζ are the linear thermal expansion coefficients, E_η, E_θ and E_ζ are Young's modulus, U_η, U_θ and U_ζ are dimensional displacement components along η-, θ- and ζ-directions, respectively.

If the body forces are absent, the equation of equilibrium for a circular conical shell along the η-direction can be written as

$$\begin{aligned}
&\frac{E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})}{A}\frac{\partial^2 U_\eta}{\partial\eta^2} + \frac{E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})}{A}\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\eta}{\partial\eta} - \frac{E_\theta(1 - v_{\eta\zeta}v_{\eta\zeta})}{A}\frac{\sin^2\alpha}{(R_1 + \eta\sin\alpha + \zeta\cos\alpha)^2} \\
&\quad\times U_\eta + G_{\eta\zeta}\frac{\partial^2 U_\eta}{\partial\zeta^2} + G_{\eta\zeta}\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\eta}{\partial\zeta} + \left[G_{\eta\zeta} + \frac{E_\zeta}{A}(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta})\right]\frac{\partial^2 U_\zeta}{\partial\eta\partial\zeta} \\
&\quad + \left[\frac{E_\zeta}{A}(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta}) - \frac{E_\zeta}{A}(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})\right]\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\zeta}{\partial\zeta} \\
&\quad + \left[G_{\eta\zeta} + \frac{E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})}{A}\right]\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\zeta}{\partial\eta} - \frac{E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta})}{A}\frac{\sin\alpha\cos\alpha}{(R_1 + \eta\sin\alpha + \zeta\cos\alpha)^2}U_\zeta \\
&\quad - \beta_\eta\frac{\partial\overline{\Theta}}{\partial\eta} - (\beta_\eta - \beta_\theta)\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\overline{\Theta} = 0
\end{aligned} \tag{2}$$

where $G_{\eta\zeta}$ is the shear modulus. If the body forces are absent, the equation of equilibrium for a circular conical shell along the ζ-direction can be written as

$$\begin{aligned}
&\frac{E_\zeta(1 - v_{\eta\theta}v_{\theta\eta})}{A}\frac{\partial^2 U_\zeta}{\partial\zeta^2} + \frac{E_\zeta(1 - v_{\eta\theta}v_{\theta\eta})}{A}\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\zeta}{\partial\zeta} - \frac{E_\theta(1 - v_{\eta\zeta}v_{\eta\zeta})}{A}\frac{\cos^2\alpha}{(R_1 + \eta\sin\alpha + \zeta\cos\alpha)^2} \\
&\quad\times U_\zeta + G_{\eta\zeta}\frac{\partial^2 U_\zeta}{\partial\eta^2} + G_{\eta\zeta}\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\zeta}{\partial\eta} + \left[G_{\eta\zeta} + \frac{E_\zeta}{A}(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta})\right]\frac{\partial^2 U_\eta}{\partial\eta\partial\zeta} \\
&\quad + \left[\frac{E_\zeta}{A}(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta}) - \frac{E_\theta}{A}(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})\right]\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\eta}{\partial\eta} + \left[G_{\eta\zeta} + \frac{E_\zeta(v_{\theta\zeta} + v_{\theta\zeta}v_{\eta\zeta})}{A}\right] \\
&\quad\times\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\frac{\partial U_\eta}{\partial\zeta} - \frac{E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta})}{A}\frac{\sin\alpha\cos\alpha}{(R_1 + \eta\sin\alpha + \zeta\cos\alpha)^2}U_\eta \\
&\quad - \beta_\zeta\frac{\partial\overline{\Theta}}{\partial\zeta} - (\beta_\zeta - \beta_\theta)\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}\overline{\Theta} = 0
\end{aligned} \tag{3}$$

The stress–displacement relations for the kth layer are

$$\begin{aligned}
\sigma^*_{\eta k} = \frac{1}{A}\bigg[&E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})\frac{\partial U_\eta}{\partial\eta} + E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}U_\eta \\
&+ E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}U_\zeta + E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta})\frac{\partial U_\zeta}{\partial\zeta}\bigg] - \beta_\eta\overline{\Theta}
\end{aligned} \tag{4}$$

$$\begin{aligned}
\sigma^*_{\theta k} = \frac{1}{A}\bigg[&E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})\frac{\partial U_\eta}{\partial\eta} + E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta})\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}U_\eta \\
&+ E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta})\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}U_\zeta + E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})\frac{\partial U_\zeta}{\partial\zeta}\bigg] - \beta_\theta\overline{\Theta}
\end{aligned} \tag{5}$$

$$\sigma^*_{\zeta k} = \frac{1}{A}\left[E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta})\frac{\partial U_\eta}{\partial \eta} + E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})\frac{\sin\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}U_\eta \right.$$
$$\left. + E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})\frac{\cos\alpha}{R_1 + \eta\sin\alpha + \zeta\cos\alpha}U_\zeta + E_\zeta(1 - v_{\eta\theta}v_{\theta\eta})\frac{\partial U_\zeta}{\partial \zeta}\right] - \beta_\zeta\overline{\Theta} \tag{6}$$

$$\tau^*_{\eta\zeta k} = G_{\eta\zeta}\left(\frac{\partial U_\zeta}{\partial \eta} + \frac{\partial U_\eta}{\partial \zeta}\right) \tag{7}$$

where $\sigma^*_{\eta k}$, $\sigma^*_{\theta k}$ and $\sigma^*_{\zeta k}$ are the dimensional normal stresses for the kth layer; $\tau^*_{\eta\zeta k}$ is the dimensional shear stress for the kth layer.

Let the boundary surface of the laminated circular conical shells be subjected to constant boundary temperatures and traction free, the boundary conditions are

$$\begin{array}{lll} \sigma^*_\eta(\eta,\zeta,\tau) = 0, & \Theta_1 = \Theta_0 + f_1 & \text{at } \eta = R_1/\sin\alpha \\ \sigma^*_\eta(\eta,\zeta,\tau) = 0, & \Theta_2 = \Theta_0 + f_2 & \text{at } \eta = L + R_1/\sin\alpha \\ \sigma^*_\zeta(\eta,\zeta,\tau) = 0, & \Theta_3 = \Theta_0 + f_3 & \text{at } \zeta = R_1/\cos\alpha \\ \sigma^*_\zeta(\eta,\zeta,\tau) = 0, & \Theta_4 = \Theta_0 + f_4 & \text{at } \zeta = (R_2 - R_1)/\cos\alpha \end{array}$$

The initial conditions are $U = 0$, and $\overline{\Theta} = 0$ at $\tau = 0$.

At the interface between two adjacent layers, the interface conditions are

$$\left.\begin{array}{ll} U_k(\eta,\tau) = U_{k+1}(\eta,\tau) & \eta = (R_1/\sin\alpha)_{k+1} \\ \sigma^*_{\eta k}(\eta,\tau) = \sigma^*_{\eta k+1}(\eta,\tau) & \eta = (R_1/\sin\alpha)_{k+1} \\ q_k(\eta,\tau) = q_{k+1}(\eta,\tau) & \eta = (R_1/\sin\alpha)_{k+1} \\ \overline{\Theta}_k(\eta,\tau) = \overline{\Theta}_{k+1}(\eta,\tau) & \eta = (R_1/\sin\alpha)_{k+1} \end{array}\right. \quad k = 2, 3, \ldots, m-1 \text{ layer}$$

where q_k is the heat flux per unit area per unit time for the kth layer.

The non-dimensional variables for the axisymmetric circular multilayered conical shells are defined as follows:

$$x = \eta(\sin\alpha/R_1), \quad z = \zeta(\cos\alpha/R_1)$$

$$T = (\Theta - \Theta_0)/\Theta_0 = \overline{\Theta}/\Theta_0, \quad u_x = U_\eta\left(\frac{\beta_\eta}{\rho C_v}\right)_k \Big/ \left(\frac{R_1}{\sin\alpha}\right)$$

$$u_z = U_\zeta\left(\frac{\beta_\eta}{\rho C_v}\right)_k \Big/ \left(\frac{R_1}{\cos\alpha}\right), \quad t = \tau\left(\frac{k_\eta}{\rho C_v}\right)_1 \Big/ \left(\frac{R_1}{\sin\alpha}\right)^2$$

$$a_{1k} = \left(\frac{k_\eta}{\rho C_v}\right)_k \Big/ \left(\frac{k_\eta}{\rho C_v}\right)_1, \quad a_{2k} = \left(\frac{k_\zeta}{\rho C_v}\right)_k \Big/ \left(\frac{k_\eta}{\rho C_v}\right)_1$$

$$a_{3k} = \left(\frac{\beta_\theta}{\rho C_v}\right)_k \Big/ \left(\frac{\beta_\eta}{\rho C_v}\right)_k, \quad a_{4k} = \left(\frac{\beta_\zeta}{\rho C_v}\right)_k \Big/ \left(\frac{\beta_\eta}{\rho C_v}\right)_k$$

$$b_{1k} = E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta})/[E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})], \quad b_{2k} = AG_{\eta\zeta}\left(\frac{\cos\alpha}{\sin\alpha}\right)^2 \Big/ [E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})]$$

$$b_{3k} = E_\zeta[(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta}) - (v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})]/[E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})]$$

$$b_{4k} = [E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta}) + AG_{\eta\zeta}]/[E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})]$$

$$b_{5k} = [E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta}) + AG_{\eta\zeta}]/[E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})]$$

$$b_{6k} = A\Theta_0\beta_\eta\left(\frac{\beta_\eta}{\rho C_v}\right)_k \Big/ [E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})], \quad b_{7k} = A\Theta_0(\beta_\eta - \beta_\theta)\left(\frac{\beta_\eta}{\rho C_v}\right)_k \Big/ [E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta})]$$

$$c_{1k} = A\left(\frac{\cos\alpha}{\sin\alpha}\right)^2 \Big/ [E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta})G_{\eta\zeta}], \quad c_{2k} = A\left(\frac{\cos\alpha}{\sin\alpha}\right)^2 \Big/ [E_\zeta(1 - v_{\eta\theta}v_{\theta\eta})G_{\eta\zeta}]$$

$$c_{3k} = \frac{AG_{\eta\zeta} + E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta})}{AG_{\eta\zeta}}\left(\frac{\cos\alpha}{\sin\alpha}\right)^2$$

$$c_{4k} = [E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta}) - E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta})]\left(\frac{\cos\alpha}{\sin\alpha}\right)^2 \Big/ AG_{\eta\zeta}$$

$$c_{5k} = \frac{AG_{\eta\zeta} + E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta})}{AG_{\eta\zeta}}\left(\frac{\cos\alpha}{\sin\alpha}\right)^2, \quad c_{6k} = \Theta_0\beta_\zeta\left(\frac{\beta_\eta}{\rho C_v}\right)_k\left(\frac{\cos\alpha}{\sin\alpha}\right)^2 \Big/ G_{\eta\zeta}$$

$$c_{7k} = \Theta_0(\beta_\zeta - \beta_\theta)\left(\frac{\beta_\eta}{\rho C_v}\right)_k\left(\frac{\cos\alpha}{\sin\alpha}\right)^2 \Big/ G_{\eta\zeta}, \quad {}_1Q_k = E_\eta(1 - v_{\theta\zeta}v_{\zeta\theta}) \Big/ \left[A\beta_{\eta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right]$$

$${}_2Q_k = E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta}) \Big/ \left[A\beta_{\eta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right]$$

$${}_3Q_k = E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta}) \Big/ \left[A\beta_{\eta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right], \quad {}_4Q_k = \beta_{\eta k}/\beta_{\eta1}$$

$${}_1R_k = E_\theta(v_{\eta\theta} + v_{\zeta\theta}v_{\eta\zeta}) \Big/ \left[A\beta_{\theta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right], \quad {}_2R_k = E_\theta(1 - v_{\eta\zeta}v_{\zeta\eta}) \Big/ \left[A\beta_{\theta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right]$$

$${}_3R_k = E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta}) \Big/ \left[A\beta_{\theta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right], \quad {}_4R_k = \beta_{\theta k}/\beta_{\theta1}$$

$${}_1P_k = E_\zeta(v_{\eta\zeta} + v_{\eta\theta}v_{\theta\zeta}) \Big/ \left[A\beta_{\zeta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right], \quad {}_2P_k = E_\zeta(v_{\theta\zeta} + v_{\theta\eta}v_{\eta\zeta}) \Big/ \left[A\beta_{\zeta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right]$$

$${}_3P_k = E_\zeta(1 - v_{\eta\theta}v_{\theta\eta}) \Big/ \left[A\beta_{\zeta1}\left(\frac{\beta_\eta}{\rho C_v}\right)_k\Theta_0\right], \quad {}_4P_k = \beta_{\zeta k}/\beta_{\zeta1}$$

$$\tau_{xz} = \tau^*_{\eta\zeta}/G_{\eta\zeta1}, \quad \sigma_{zk} = \sigma^*_{\zeta k}/(\beta_{\zeta1}\Theta_0), \quad \sigma_{\theta k} = \sigma^*_{\theta k}/(\beta_{\theta1}\Theta_0), \quad \sigma_{xk} = \sigma^*_{\eta k}/(\beta_{\eta1}\Theta_0)$$

$${}_1S_k = \left(\frac{G_{\eta\zeta}}{G_{\eta\zeta1}}\right)\left(\frac{\cos\alpha}{\sin\alpha}\right) \Big/ \left(\frac{\beta_\eta}{\rho C_v}\right)_k, \quad {}_2S_k = \left(\frac{G_{\eta\zeta}}{G_{\eta\zeta1}}\right)\left(\frac{\sin\alpha}{\cos\alpha}\right) \Big/ \left(\frac{\beta_\eta}{\rho C_v}\right)_k$$

where x is the non-dimensional meridional direction; z is the non-dimensional normal direction; T is the non-dimensional temperature; u_x and u_z are the non-dimensional displacement components; σ_{xk}, $\sigma_{\theta k}$ and σ_{zk} are the non-dimensional normal stresses for the kth layer; τ_{xz} is the non-dimensional shear stress for the kth layer.

3. Computational procedures

By substituting the non-dimensional quantities into the governing equations (1)–(3), and stress–displacement relations (4)–(7), we arrive at the following non-dimensional equations:

$$\left\{a_{1k}\frac{\partial^2}{\partial x^2}+\frac{a_{1k}}{1+x+z}\frac{\partial}{\partial x}+a_{2k}\frac{\partial^2}{\partial z^2}+\frac{a_{2k}}{1+x+z}\frac{\partial}{\partial z}\right\}T$$
$$=\frac{\partial T}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial t}\right)+\frac{a_{3k}}{1+x+z}\left(\frac{\partial u_x}{\partial t}\right)+\frac{a_{3k}}{1+x+z}\left(\frac{\partial u_z}{\partial t}\right)+a_{4k}\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial t}\right) \tag{8}$$

$$\frac{\partial^2 u_x}{\partial x^2}+\frac{1}{1+x+z}\frac{\partial u_x}{\partial x}-\frac{b_{1k}}{(1+x+z)^2}u_x+b_{2k}\frac{\partial^2 u_x}{\partial z^2}+\frac{b_{2k}}{1+x+z}\frac{\partial u_x}{\partial z}+\frac{b_{3k}}{1+x+z}\frac{\partial u_z}{\partial z}-\frac{b_{1k}}{(1+x+z)^2}u_z$$
$$+\frac{b_{4k}}{1+x+z}\frac{\partial u_z}{\partial x}+b_{5k}\frac{\partial^2 u_z}{\partial x\partial z}-b_{6k}\frac{\partial T}{\partial x}-\frac{b_{7k}}{1+x+z}T=0 \tag{9}$$

$$\frac{\partial^2 u_z}{\partial x^2}+\frac{1}{1+x+z}\frac{\partial u_z}{\partial x}-\frac{c_{1k}}{(1+x+z)^2}u_z+c_{2k}\frac{\partial^2 u_z}{\partial z^2}+\frac{c_{2k}}{1+x+z}\frac{\partial u_z}{\partial z}+\frac{c_{3k}}{1+x+z}\frac{\partial u_x}{\partial z}-\frac{c_{1k}}{(1+x+z)^2}u_x$$
$$+\frac{c_{4k}}{1+x+z}\frac{\partial u_x}{\partial x}+c_{5k}\frac{\partial^2 u_x}{\partial x\partial z}-c_{6k}\frac{\partial T}{\partial z}-\frac{c_{7k}}{1+x+z}T=0 \tag{10}$$

$$\sigma_{xk}={}_1Q_k\frac{\partial u_x}{\partial x}+{}_2Q_k\frac{u_x}{1+x+z}+{}_2Q_k\frac{u_z}{1+x+z}+{}_3Q_k\frac{\partial u_z}{\partial z}-{}_4Q_kT \tag{11}$$

$$\sigma_{\theta k}={}_1R_k\frac{\partial u_x}{\partial x}+{}_2R_k\frac{u_x}{1+x+z}+{}_2R_k\frac{u_z}{1+x+z}+{}_3R_k\frac{\partial u_z}{\partial z}-{}_4R_kT \tag{12}$$

$$\sigma_{zk}={}_1P_k\frac{\partial u_x}{\partial x}+{}_2P_k\frac{u_x}{1+x+z}+{}_2P_k\frac{u_z}{1+x+z}+{}_3P_k\frac{\partial u_z}{\partial z}-{}_4P_kT \tag{13}$$

$$\tau_{xzk}={}_1S_k\frac{\partial u_x}{\partial z}+{}_2S_k\frac{\partial u_z}{\partial x} \tag{14}$$

The non-dimensional boundary conditions can be written as

$$\begin{aligned}
&\sigma_x(x,z,t)=0, \quad T_1=f_1/\Theta_0 \quad &&\text{at } x=x_{\text{top}}\\
&\sigma_x(x,z,t)=0, \quad T_2=f_2/\Theta_0 \quad &&\text{at } x=x_{\text{bottom}}\\
&\sigma_z(x,z,t)=0, \quad T_3=f_3/\Theta_0 \quad &&\text{at } z=z_{\text{inner}}\\
&\sigma_z(x,z,t)=0, \quad T_4=f_4/\Theta_0 \quad &&\text{at } z=z_{\text{outer}}
\end{aligned}$$

By applying the central difference scheme in Eqs. (8)–(10), the following discretized equations are obtained:

$$a_{1k}\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{(\Delta x)^2}+a_{1k}\frac{1}{1+x_{i,j}+z_{i,j}}\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta x}+a_{2k}\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{(\Delta z)^2}a_{2k}\frac{1}{1+x_{i,j}+z_{i,j}}$$
$$\times\frac{T_{i,j+1}-T_{i,j-1}}{2\Delta z}=\frac{\partial T_{i,j}}{\partial t}+\frac{\left(\frac{\partial u_x}{\partial t}\right)_{i+1,j}-\left(\frac{\partial u_x}{\partial t}\right)_{i-1,j}}{2\Delta x}+\frac{a_{3k}}{1+x_{i,j}+z_{i,j}}\left(\frac{\partial u_x}{\partial t}\right)_{i,j}+\frac{a_{3k}}{1+x_{i,j}+z_{i,j}}\left(\frac{\partial u_z}{\partial t}\right)_{i,j}$$
$$+a_{4k}\frac{\left(\frac{\partial u_z}{\partial t}\right)_{i,j+1}-\left(\frac{\partial u_z}{\partial t}\right)_{i,j-1}}{2\Delta z} \tag{15}$$

$$\begin{aligned}
&\frac{u_{xi+1,j}-2u_{xi,j}+u_{xi-1,j}}{(\Delta x)^2}+\frac{1}{1+x_{i,j}+z_{i,j}}\frac{u_{xi+1,j}-u_{xi-1,j}}{2\Delta x}-\frac{b_{1k}}{(1+x_{i,j}+z_{i,j})^2}u_{xi,j}\\
&\quad+b_{2k}\frac{u_{xi,j+1}-2u_{xi,j}+u_{xi,j-1}}{(\Delta z)^2}+\frac{b_{2k}}{1+x_{i,j}+z_{i,j}}\frac{u_{xi,j+1}-u_{xi,j-1}}{2\Delta z}+\frac{b_{3k}}{1+x_{i,j}+z_{i,j}}\frac{u_{zi,j+1}-u_{zi,j-1}}{2\Delta z}\\
&\quad-\frac{b_{1k}}{(1+x_{i,j}+z_{i,j})^2}u_{zi,j}+\frac{b_{4k}}{1+x_{i,j}+z_{i,j}}\frac{u_{zi+1,j}-u_{zi-1,j}}{2\Delta x}+b_{5k}\frac{u_{zi+1,j+1}-u_{zi+1,j-1}-u_{zi-1,j+1}+u_{zi-1,j-1}}{4\Delta x\Delta z}\\
&\quad-b_{6k}\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta x}-\frac{b_{7k}}{1+x_{i,j}+z_{i,j}}T_{i,j}=0
\end{aligned}\tag{16}$$

$$\begin{aligned}
&\frac{u_{zi+1,j}-2u_{zi,j}+u_{zi-1,j}}{(\Delta x)^2}+\frac{1}{1+x_{i,j}+z_{i,j}}\frac{u_{zi+1,j}-u_{zi-1,j}}{2\Delta x}-\frac{c_{1k}}{(1+x_{i,j}+z_{i,j})^2}u_{zi,j}\\
&\quad+c_{2k}\frac{u_{zi,j+1}-2u_{zi,j}+u_{zi,j-1}}{(\Delta z)^2}+\frac{c_{2k}}{1+x_{i,j}+z_{i,j}}\frac{u_{zi,j+1}-u_{zi,j-1}}{2\Delta z}+\frac{c_{3k}}{1+x_{i,j}+z_{i,j}}\frac{u_{xi,j+1}-u_{xi,j-1}}{2\Delta z}\\
&\quad-\frac{c_{1k}}{(1+x_{i,j}+z_{i,j})^2}u_{xi,j}+\frac{c_{4k}}{1+x_{i,j}+z_{i,j}}\frac{u_{xi+1,j}-u_{xi-1,j}}{2\Delta x}+c_{5k}\frac{u_{xi+1,j+1}-u_{xi+1,j-1}-u_{xi-1,j+1}+u_{xi-1,j-1}}{4\Delta x\Delta z}\\
&\quad-c_{6k}\frac{T_{i,j+1}-T_{i,j-1}}{2\Delta z}-\frac{c_{7k}}{1+x_{i,j}+z_{i,j}}T_{i,j}=0
\end{aligned}\tag{17}$$

By applying the central difference in the stress–displacement relations (11)–(14), the following discretized equations are obtained:

$$\sigma_{xk}={}_1Q_k\frac{u_{xi+1,j}-u_{xi-1,j}}{2\Delta x}+{}_2Q_k\frac{u_{xi,j}}{1+x_{i,j}+z_{i,j}}+{}_2Q_k\frac{u_{zi,j}}{1+x_{i,j}+z_{i,j}}+{}_3Q_k\frac{u_{zi,j+1}-u_{zi,j-1}}{2\Delta z}-{}_4Q_kT_{i,j}\tag{18}$$

$$\sigma_{\theta k}={}_1R_k\frac{u_{xi+1,j}-u_{xi-1,j}}{2\Delta x}+{}_2R_k\frac{u_{xi,j}}{1+x_{i,j}+z_{i,j}}+{}_2R_k\frac{u_{zi,j}}{1+x_{i,j}+z_{i,j}}+{}_3R_k\frac{u_{zi,j+1}-u_{zi,j-1}}{2\Delta z}-{}_4R_kT_{i,j}\tag{19}$$

$$\sigma_{zk}={}_1P_k\frac{u_{xi+1,j}-u_{xi-1,j}}{2\Delta x}+{}_2P_k\frac{u_{xi,j}}{1+x_{i,j}+z_{i,j}}+{}_2P_k\frac{u_{zi,j}}{1+x_{i,j}+z_{i,j}}+{}_3P_k\frac{u_{zi,j+1}-u_{zi,j-1}}{2\Delta z}-{}_4P_kT_{i,j}\tag{20}$$

$$\tau_{xzk}={}_1S_k\frac{u_{xi,j+1}-u_{xi,j-1}}{2\Delta z}+{}_2S_k\frac{u_{zi+1,j}-u_{zi-1,j}}{2\Delta x}\tag{21}$$

The Laplace transform of a function $\Phi(t)$ and its inverse are defined by

$$\overline{\Phi}(s)=L[\Phi(t)]=\int_0^\infty e^{-st}\Phi(t)\,dt$$

$$\Phi(t)=L^{-1}[\overline{\Phi}(s)]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}\overline{\Phi}(s)\,ds$$

By taking the Laplace transform for Eqs. (15)–(21), the following equations are obtained:

$$a_{1k}\frac{\overline{T}_{i+1,j}-2\overline{T}_{i,j}+\overline{T}_{i-1,j}}{(\Delta x)^2}+a_{1k}\frac{1}{1+x_{i,j}+z_{i,j}}\frac{\overline{T}_{i+1,j}-\overline{T}_{i-1,j}}{2\Delta x}+a_{2k}\frac{\overline{T}_{i,j+1}-2\overline{T}_{i,j}+\overline{T}_{i,j-1}}{(\Delta z)^2}a_{2k}$$
$$\times\frac{1}{1+x_{i,j}+z_{i,j}}\frac{\overline{T}_{i,j+1}-\overline{T}_{i,j-1}}{2\Delta z}$$
$$=(T_{i,j_{\rm in}}+s\overline{T}_{i,j})+\frac{(u_{x{\rm in}}+s\bar{u}_x)_{i+1,j}-(u_{x{\rm in}}+s\bar{u}_x)_{i-1,j}}{2\Delta x}+\frac{a_{3k}}{1+x_{i,j}+z_{i,j}}(u_{xi,j_{\rm in}}+s\bar{u}_{xi,j})$$
$$+\frac{a_{3k}}{1+x_{i,j}+z_{i,j}}(u_{zi,j_{\rm in}}+s\bar{u}_{zi,j})+a_{4k}\frac{(u_{z{\rm in}}+s\bar{u}_z)_{i,j+1}-(u_{z{\rm in}}+s\bar{u}_z)_{i,j-1}}{2\Delta z} \quad (22)$$

$$\frac{\bar{u}_{xi+1,j}-2\bar{u}_{xi,j}+\bar{u}_{xi-1,j}}{(\Delta x)^2}+\frac{1}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{xi+1,j}-\bar{u}_{xi-1,j}}{2\Delta x}-\frac{b_{1k}}{(1+x_{i,j}+z_{i,j})^2}\bar{u}_{xi,j}$$
$$+b_{2k}\frac{\bar{u}_{xi,j+1}-2\bar{u}_{xi,j}+\bar{u}_{xi,j-1}}{(\Delta z)^2}+\frac{b_{2k}}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{xi,j+1}-\bar{u}_{xi,j-1}}{2\Delta z}+\frac{b_{3k}}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{zi,j+1}-\bar{u}_{zi,j-1}}{2\Delta z}$$
$$-\frac{b_{1k}}{(1+x_{i,j}+z_{i,j})^2}\bar{u}_{zi,j}+\frac{b_{4k}}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{zi+1,j}-\bar{u}_{zi-1,j}}{2\Delta x}+b_{5k}\frac{\bar{u}_{zi+1,j+1}-\bar{u}_{zi+1,j-1}-\bar{u}_{zi-1,j+1}+\bar{u}_{zi-1,j-1}}{4\Delta x\Delta z}$$
$$-b_{6k}\frac{\overline{T}_{i+1,j}-\overline{T}_{i-1,j}}{2\Delta x}-\frac{b_{7K}}{1+x_{i,j}+z_{i,j}}\overline{T}_{i,j}=0 \quad (23)$$

$$\frac{\bar{u}_{zi+1,j}-2\bar{u}_{zi,j}+\bar{u}_{zi-1,j}}{(\Delta x)^2}+\frac{1}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{zi+1,j}-\bar{u}_{zi-1,j}}{2\Delta x}-\frac{c_{1k}}{(1+x_{i,j}+z_{i,j})^2}\bar{u}_{zi,j}$$
$$+c_{2k}\frac{\bar{u}_{zi,j+1}-2\bar{u}_{zi,j}+\bar{u}_{zi,j-1}}{(\Delta z)^2}+\frac{c_{2k}}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{zi,j+1}-\bar{u}_{zi,j-1}}{2\Delta z}+\frac{c_{3k}}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{xi,j+1}-\bar{u}_{xi,j-1}}{2\Delta z}$$
$$-\frac{c_{1k}}{(1+x_{i,j}+z_{i,j})^2}\bar{u}_{xi,j}+\frac{c_{4k}}{1+x_{i,j}+z_{i,j}}\frac{\bar{u}_{xi+1,j}-\bar{u}_{xi-1,j}}{2\Delta x}+c_{5k}\frac{\bar{u}_{xi+1,j+1}-\bar{u}_{xi+1,j-1}-\bar{u}_{xi-1,j+1}+\bar{u}_{xi-1,j-1}}{4\Delta x\Delta z}$$
$$-c_{6k}\frac{\overline{T}_{i,j+1}-\overline{T}_{i,j-1}}{2\Delta z}-\frac{c_{7k}}{1+x_{i,j}+z_{i,j}}\overline{T}_{i,j}=0 \quad (24)$$

$$\bar{\sigma}_{xk}={}_1Q_k\frac{\bar{u}_{xi+1,j}-\bar{u}_{xi-1,j}}{2\Delta x}+{}_2Q_k\frac{\bar{u}_{xi,j}}{1+x_{i,j}+z_{i,j}}+{}_2Q_k\frac{\bar{u}_{zi,j}}{1+x_{i,j}+z_{i,j}}+{}_3Q_k\frac{\bar{u}_{zi,j+1}-\bar{u}_{zi,j-1}}{2\Delta z}-{}_4Q_k\overline{T}_{i,j} \quad (25)$$

$$\bar{\sigma}_{\theta k}={}_1R_k\frac{\bar{u}_{xi+1,j}-\bar{u}_{xi-1,j}}{2\Delta x}+{}_2R_k\frac{\bar{u}_{xi,j}}{1+x_{i,j}+z_{i,j}}+{}_2R_k\frac{\bar{u}_{zi,j}}{1+x_{i,j}+z_{i,j}}+{}_3R_k\frac{\bar{u}_{zi,j+1}-\bar{u}_{zi,j-1}}{2\Delta z}-{}_4R_k\overline{T}_{i,j} \quad (26)$$

$$\bar{\sigma}_{zk}={}_1P_k\frac{\bar{u}_{xi+1,j}-\bar{u}_{xi-1,j}}{2\Delta x}+{}_2P_k\frac{\bar{u}_{xi,j}}{1+x_{i,j}+z_{i,j}}+{}_2P_k\frac{\bar{u}_{zi,j}}{1+x_{i,j}+z_{i,j}}+{}_3P_k\frac{\bar{u}_{zi,j+1}-\bar{u}_{zi,j-1}}{2\Delta z}-{}_4P_k\overline{T}_{i,j} \quad (27)$$

$$\bar{\tau}_{xzk}={}_1S_k\frac{\bar{u}_{xi,j+1}-\bar{u}_{xi,j-1}}{2\Delta z}+{}_2S_k\frac{\bar{u}_{zi+1,j}-\bar{u}_{zi-1,j}}{2\Delta x} \quad (28)$$

By letting the boundary surface of the laminated circular conical shells be subject to constant boundary temperatures and traction free, after taking the Laplace transform, the boundary conditions in the transformed domain become

$$\begin{array}{lll}\bar{\sigma}_x(x,z,s)=0, & \overline{T}_1=\bar{f}_1/\Theta_0 & \text{at } x=x_{\text{top}}\\ \bar{\sigma}_x(x,z,s)=0, & \overline{T}_2=\bar{f}_2/\Theta_0 & \text{at } x=x_{\text{bottom}}\\ \bar{\sigma}_z(x,z,s)=0, & \overline{T}_3=\bar{f}_3/\Theta_0 & \text{at } z=z_{\text{inner}}\\ \bar{\sigma}_z(x,z,s)=0, & \overline{T}_4=\bar{f}_4/\Theta_0 & \text{at } z=z_{\text{outer}}\end{array}$$

and the interface conditions are as follows:

$$\left.\begin{array}{ll}\bar{u}_k(x,s)=\bar{u}_{k+1}(x,s) & x=(R_1/\sin\alpha)_{k+1}\\ \bar{\sigma}_{xk}(x,s)=\bar{\sigma}_{xk+1}(x,s) & x=(R_1/\sin\alpha)_{k+1}\\ \bar{q}_k(x,s)=\bar{q}_{k+1}(x,s) & x=(R_1/\sin\alpha)_{k+1}\\ \overline{T}_k(x,s)=\overline{T}_{k+1}(x,s) & x=(R_1/\sin\alpha)_{k+1}\end{array}\right. \quad k=2,3,\ldots,m-1 \text{ layer}$$

By using boundary conditions and interface conditions in Eqs. (22)–(24), the following equation in matrix form is obtained:

$$\{[M_1]-s[I]\}\{\overline{T}_{ij}\}+s[M_2]\{\bar{u}_{zij}\}+s[M_3]\{\bar{u}_{xij}\}=\frac{1}{s}[M_4] \tag{29}$$

$$[M_5]\{\overline{T}_{ij}\}+[M_6]\{\bar{u}_{zij}\}+[M_7]\{\bar{u}_{xij}\}=0 \tag{30}$$

$$[M_8]\{\overline{T}_{ij}\}+[M_9]\{\bar{u}_{zij}\}+[M_{10}]\{\bar{u}_{xij}\}=0 \tag{31}$$

where the matrices $[M_1]$ to $[M_{10}]$ are given in Appendix A.

By solving Eqs. (30) and (31), $\bar{u}_{xij}$ and $\bar{u}_{zij}$ can be obtained as

$$\{\bar{u}_{xij}\}=([M_{10}]-[M_9][M_6]^{-1}[M_7])^{-1}([M_9][M_6]^{-1}[M_5]-[M_8])\{\overline{T}_{ij}\} \tag{32}$$

$$\{\bar{u}_{zij}\}=([M_9]-[M_{10}][M_7]^{-1}[M_6])^{-1}([M_{10}][M_7]^{-1}[M_5]-[M_8])\{\overline{T}_{ij}\} \tag{33}$$

By substituting Eqs. (32) and (33) into Eq. (29), one has

$$\{[M]-s[I]\}\{\overline{T}_{ij}\}=\{\overline{B}_{ij}\} \tag{34}$$

in which

$$\begin{aligned}[M]=\{&[I]-[M_2]([M_9]-[M_{10}][M_7]^{-1}[M_6])^{-1}([M_{10}][M_7]^{-1}[M_5]-[M_8])\\ &-[M_3]([M_{10}]-[M_9][M_6]^{-1}[M_7])^{-1}([M_9][M_6]^{-1}[M_5]-[M_8])\}^{-1}[M_1]\end{aligned}$$

and

$$\begin{aligned}\{\overline{B}_{ij}\}=\{&[I]-[M_2]([M_9]-[M_{10}][M_7]^{-1}[M_6])^{-1}([M_{10}][M_7]^{-1}[M_5]-[M_8])\\ &-[M_3]([M_{10}]-[M_9][M_6]^{-1}[M_7])^{-1}([M_9][M_6]^{-1}[M_5]-[M_8])\}^{-1}\frac{1}{s}[M_4]\end{aligned}$$

Since the $(N^2\times N^2)$ matrix $[M]$ is a non-singular real matrix, matrix $[M]$ possesses a set of N^2 linearly independent eigenvectors, hence matrix $[M]$ is diagonalizable. There exists a non-singular transition matrix $[P]$ such that $[P]^{-1}[M][P]=\text{diag}[M]$, that is, matrices $[M]$ and $\text{diag}[M]$ are similar, where matrix $\text{diag}[M]$ is defined as

$$\text{diag}[M]=\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{N^2}\end{bmatrix} \tag{35}$$

in which λ_j $(j=1,2,\ldots,N^2)$ is the eigenvalue of matrix $[M]$.

Table 1
Geometry and material constants of a laminated circular conical shell ($z_{\text{outer}}/z_{\text{inner}} = 1.7$, $L = 4.5$, $\alpha = \pi/6$)

	Layer 1	Layer 2	Layer 3
$E_\eta = E_\theta = E_\zeta$	50E6	58E6	50E6
$k_\eta = k_\theta = k_\zeta$	18	22	18
$\alpha_\eta = \alpha_\theta = \alpha_\zeta$	4E−6	4E−6	4E−6
$v_{\eta\theta} = v_{\theta\eta}$	0.2	0.3	0.2
$v_{\eta\zeta} = v_{\zeta\eta}$	0.3	0.2	0.1
$v_{\zeta\theta} = v_{\theta\zeta}$	0.15	0.15	0.15
$G_{\eta\zeta}$	15E6	18E6	15E6
ρ	0.095	0.095	0.095
C_v	0.3	0.3	0.3

Table 2
The temperature under different times and different number of gird points of finite difference method at $x = 3.15$, $z = 1.55$

	Girds = 36 ($N = 6$)	Girds = 64 ($N = 8$)	Girds = 100 ($N = 10$)	Girds = 144 ($N = 12$)	Girds = 196 ($N = 14$)
Time = 0.1	44.579	45.581	46.147	46.233	46.238
Time = 0.5	53.052	55.491	57.028	57.131	57.137
Time = 1.0	54.603	57.490	59.007	59.112	59.118
Time = 2.0	56.073	58.590	60.890	60.999	61.014
Time = 3.0	56.114	58.947	60.944	61.053	61.056
Time = 4.0	56.115	58.948	60.946	61.055	61.058
Time = 5.0	56.115	58.948	60.946	61.055	61.058

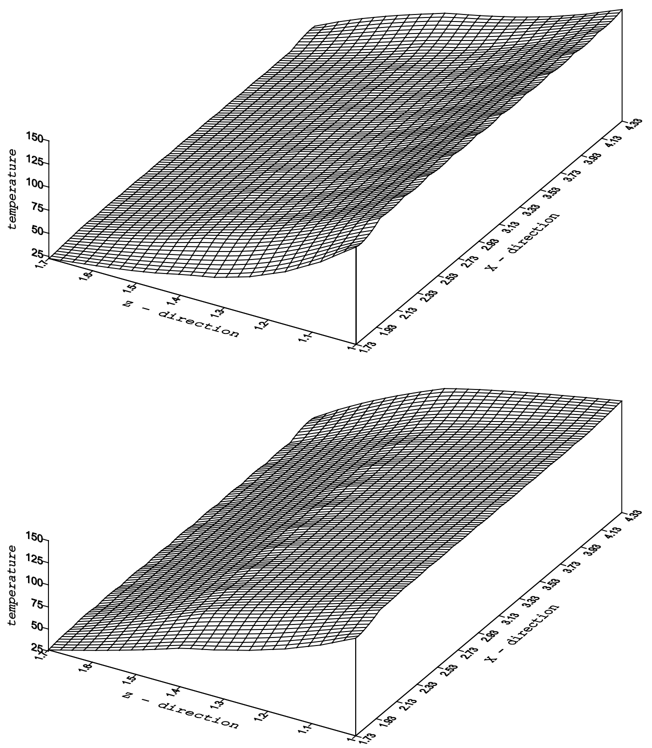

Fig. 2. Temperature distribution along z- and x-directions: (a) $t = 0.1$, (b) $t = 5$.

By substituting Eq. (35) into (34), the following matrix equation is obtained:

$$\{\mathrm{diag}[M] - s[I]\}\{\overline{T}^*_{ij}\} = \{\overline{B}^*_{ij}\} \tag{36}$$

where

$$\{\overline{T}^*_{ij}\} = [P]^{-1}\{\overline{T}_{ij}\} \quad \text{and} \quad \{\overline{B}^*_{ij}\} = [P]^{-1}\{\overline{B}_{ij}\}$$

The following solutions are obtained immediately from Eq. (36):

$$\overline{T}^*_{ij} = \overline{B}^*_{ij}/(\lambda_j - s) \tag{37}$$

By substituting Eq. (37) into (32) and (33), the equations can be obtained as following:

$$\bar{u}^*_{xij} = \overline{A}^*_{ij}/(\lambda_j - s) \tag{38}$$

$$\bar{u}^*_{zij} = \overline{D}^*_{ij}/(\lambda_j - s) \tag{39}$$

where

$$\{\bar{u}^*_{xij}\} = [P]^{-1}\{\bar{u}_{xij}\}$$

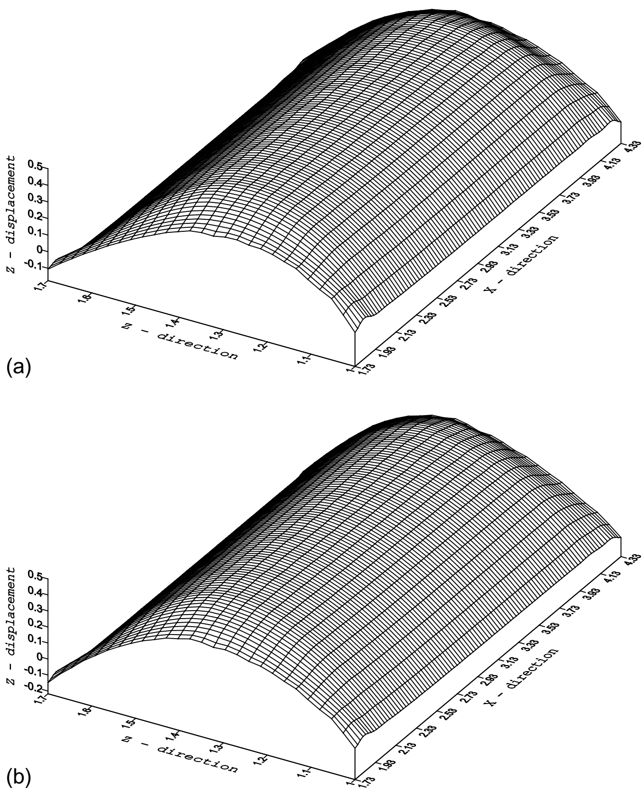

Fig. 3. Displacement component u_z along z- and x-directions: (a) $t = 0.1$ (b) $t = 5$.

$$\{\bar{u}^*_{zij}\} = [P]^{-1}\{\bar{u}_{zij}\}$$

$$\{\overline{A}^*_{ij}\} = ([M_{10}] - [M_9][M_6]^{-1}[M_7])^{-1}([M_9][M_6]^{-1}[M_5] - [M_8])[P]^{-1}\{\overline{B}_{ij}\}$$

$$\{\overline{D}^*_{ij}\} = ([M_9] - [M_{10}][M_7]^{-1}[M_6])^{-1}([M_{10}][M_7]^{-1}[M_5] - [M_8])[P]^{-1}\{\overline{B}_{ij}\}$$

By taking the inverse Laplace transform on Eqs. (37)–(39), the solutions for T^*_{ij}, u^*_{xij} and u^*_{zij} can be obtained. Substituting T^*_{ij}, u^*_{xij} and u^*_{zij} into the following relations, the temperature distribution T_{ij}, the displacements u_{xij} and u_{zij} can be obtained as:

$$\{T_{ij}\} = [P]\{T^*_{ij}\} \tag{40}$$

$$\{u_{xij}\} = [P]\{u^*_{xij}\} \tag{41}$$

$$\{u_{zij}\} = [P]\{u^*_{zij}\} \tag{42}$$

Therefore, by substituting T_{ij}, u_{zij}, and u_{xij} into Eqs. (18)–(21), the x-direction stress σ_x, the circumferential stress σ_θ, z-direction stress σ_z, and the shear stress τ_{xz} can all be obtained.

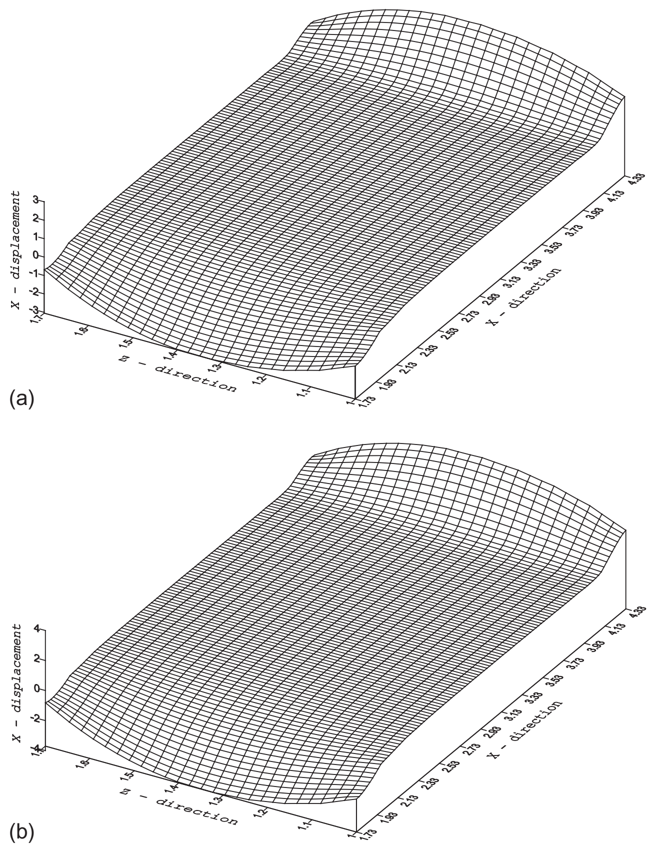

Fig. 4. Displacement component u_x along z- and x-directions: (a) $t = 0.1$, (b) $t = 5$.

4. Numerical results and discussions

In this section, some results of the temperature distribution in a laminated circular conical shell, displacement and thermal stresses are calculated numerically. To illustrate the foregoing analysis, numerical calculations for a circular multilayered conical shell under axisymmetric heating at the boundary surface were performed. The laminated circular conical shell is composed of three different isotropic layers. The geometry parameters and the material quantities of this laminated circular conical shell are shown in Table 1. The non-dimensional inner and outer radii of the cone at the small and large edges are assumed to be 1.0 and 1.7, respectively. The non-dimensional inner and outer temperatures are assumed to be 150 and 25, respectively. The top and bottom non-dimensional temperatures are assumed to be 25 and 150, respectively. The both ends are traction free. Each layer is assumed to have a different thickness h_i. In this examined case, the laminated circular conical shell composed of three layers, $h_1 = 0.3$, $h_2 = 0.1$ and $h_3 = 0.3$, and the semi-vertex angle of the cone is $\alpha = \pi/6$.

For the convergence test of the present method, we calculate the temperature values at the point $x = 3.15$, $z = 1.55$ with different number of finite difference gird points and different time as shown in Table 2. Results show that the temperature changes very small as the number of the grid point increases, say gird

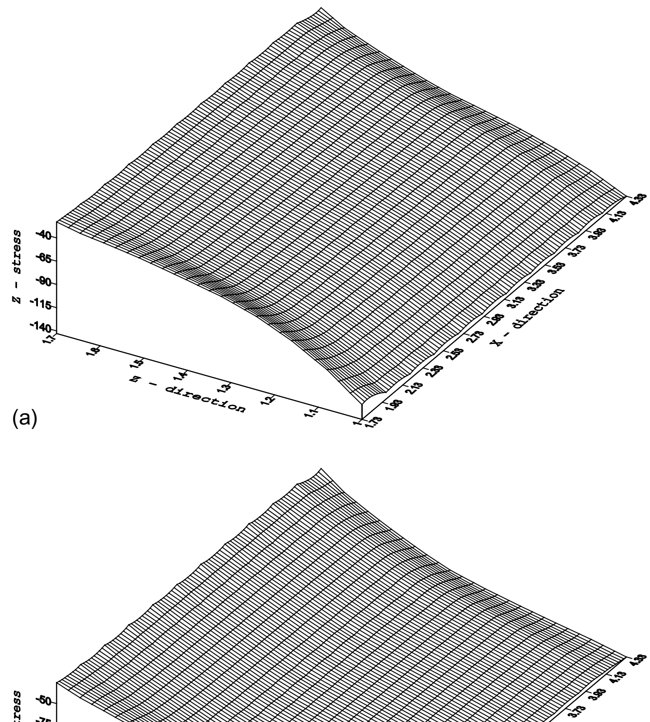

Fig. 5. Stress distribution σ_z along z- and x-directions: (a) $t = 0.1$, (b) $t = 5$.

points is 144 ($N = 12$). From Table 2, we can see that under various number of gird points of finite difference method, the solutions are rapidly convergent. It is well known that the temperature tends to steady as the time increases. As the time increases, for example, $t = 3$ to 5, the temperatures vary slightly in each layer, since the steady state is approached as the time increases. Therefore, in the present study, we chose gird points = 196 ($N = 14$) to evaluate the temperature distribution, displacement and thermal stresses in a laminated circular conical shell for small time $t = 0.1$, which is in transient state, and large time $t = 5.0$, which approaches the steady state.

Fig. 2(a) and (b) show the temperature distribution along the z- and x-directions of the laminated circular conical shell at $t = 0.1$ and 5, respectively. The temperature gradient varies in each layer because of the difference in thermal conductivity coefficients. In both cases, it is observed that the temperature distribution across each layer is generally curved at short time intervals. As the time interval becomes larger, say, $t = 5$, the temperature distribution changes slightly.

Fig. 3(a) and (b) show the variation of displacement in z-direction u_z along the z- and x-directions for the laminated circular conical shell at $t = 0.1$ and 5, respectively. From these figures, the locations of the points of maximum z-direction displacement u_z along the z-direction occur at the center nearly. As the increasing

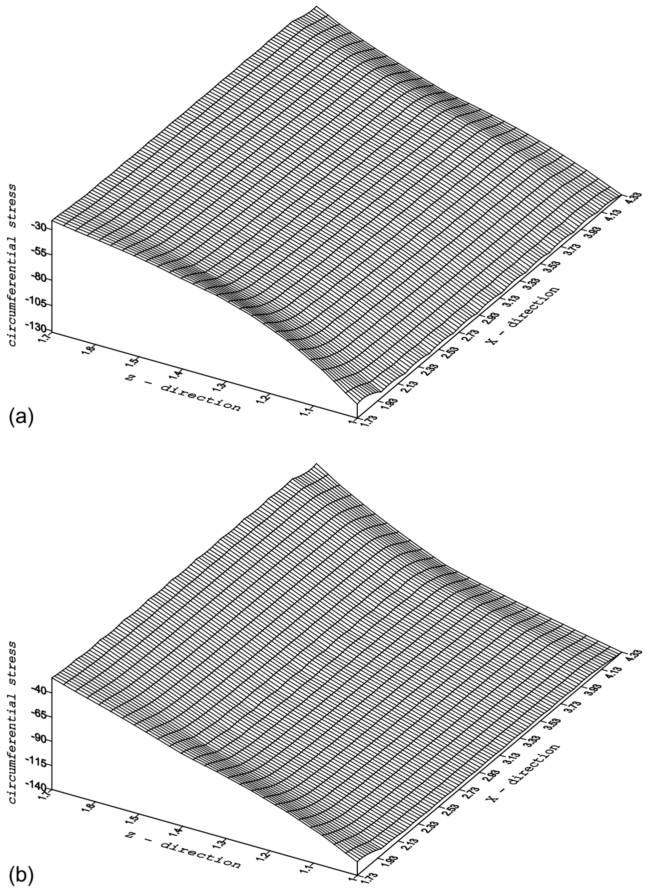

Fig. 6. Circumferential stress along z- and x-directions: (a) $t = 0.1$, (b) $t = 5$.

of the time, the locations of the points of maximum displacement move to the inner lateral surface of the laminated circular conical shells. It is noted that the displacement in z-direction has negative values, which is due to the thermal expansion and the choice of coordinates ($u_x = 0$ at $x = x_{\mathrm{top}}$). For a large t, say, $t = 5$, the larger negative value of u_z is obtained as would be expected. Fig. 4(a) and (b) show the x-direction displacement varying in the z- and x-directions of the laminated circular conical shells at $t = 0.1$ and 5, respectively. The locations of the points of maximum x-direction displacement u_x along the z-direction occur at the center. Due to the difference between the top and bottom temperatures, the x-direction displacement u_x is increasing from the top end to the bottom end. It is noted that the displacement in x-direction has negative values, which is due to the thermal expansion and the coordinates is chosen as $u_x = 0$ in the midplane.

Fig. 5(a) and (b) show the thermal stress distribution σ_z along the z- and x-directions at $t = 0.1$ and $t = 5$, respectively. From these figures, the locations of the points of maximum stress σ_z occur at the inner surfaces (at $z = 1$). The larger the surrounding temperatures will have the greater the thermal stress σ_z. As time increases to 5, the thermal stress arrives at its steady state value, which is greater than its transient state value as shown in Fig. 5.

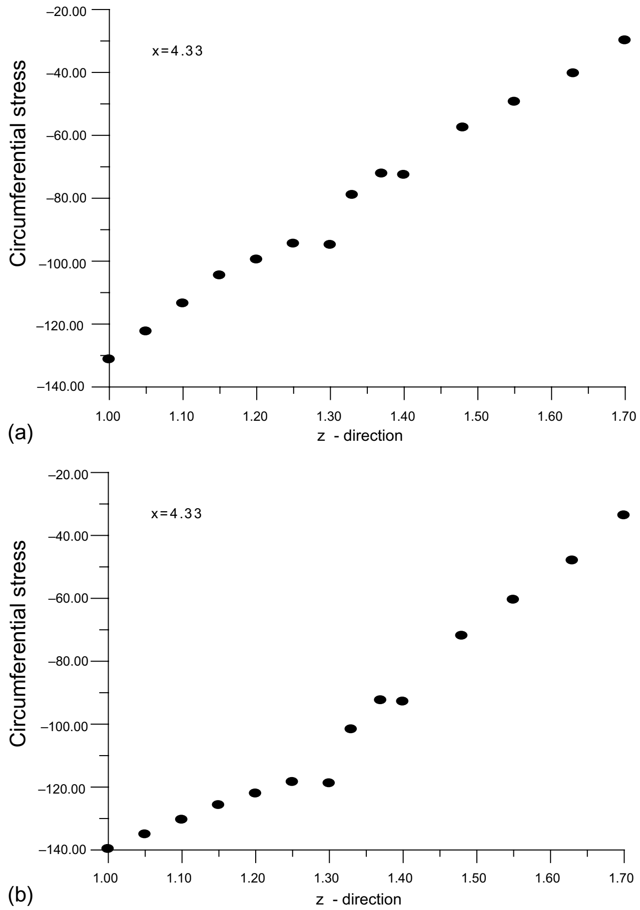

Fig. 7. Circumferential stress along z- and x-directions at $x = 4.33$: (a) $t = 0.1$, (b) $t = 5$.

Fig. 6(a) and (b) show the circumferential stress σ_θ along the z- and x-directions of the laminated circular conical shell at $t = 0.5$ and $t = 5$, respectively. Due to the curve-fitting feature of the three-dimensional pictures of the software package Surfer, it is difficult to see the discontinuity of the circumferential thermal stress. In order to reveal the discontinuity of the thermal stress, Fig. 7(a) and (b) show the circumferential stress σ_θ along the z-directions at $x = 4.33$. It is noted that the circumferential stress σ_θ has discontinuity at the interfaces as would be expected.

Fig. 8(a) and (b) show the stress distribution σ_x along the z- and x-directions of the laminated circular conical shell at $t = 0.1$ and $t = 5$, respectively. The locations of the points of maximum stress σ_x occur at the inner surfaces. From Figs. 5, 6 and 8, it is noted that the thermal stress distribution σ_z is larger than other thermal stress components. Fig. 9(a) and (b) show that the distribution of the shear stress τ_{xz} in the laminated circular conical shell at $t = 0.1$ and $t = 5$, respectively. The shear stress τ_{xz} is very small as compared to other thermal stress components.

The above discussions demonstrate that the present method for the conical coordinates can obtain stable solutions at a specific time; thus it is a powerful and efficient method for solving the coupled transient thermoelastic problems of a circular multilayered conical shell.

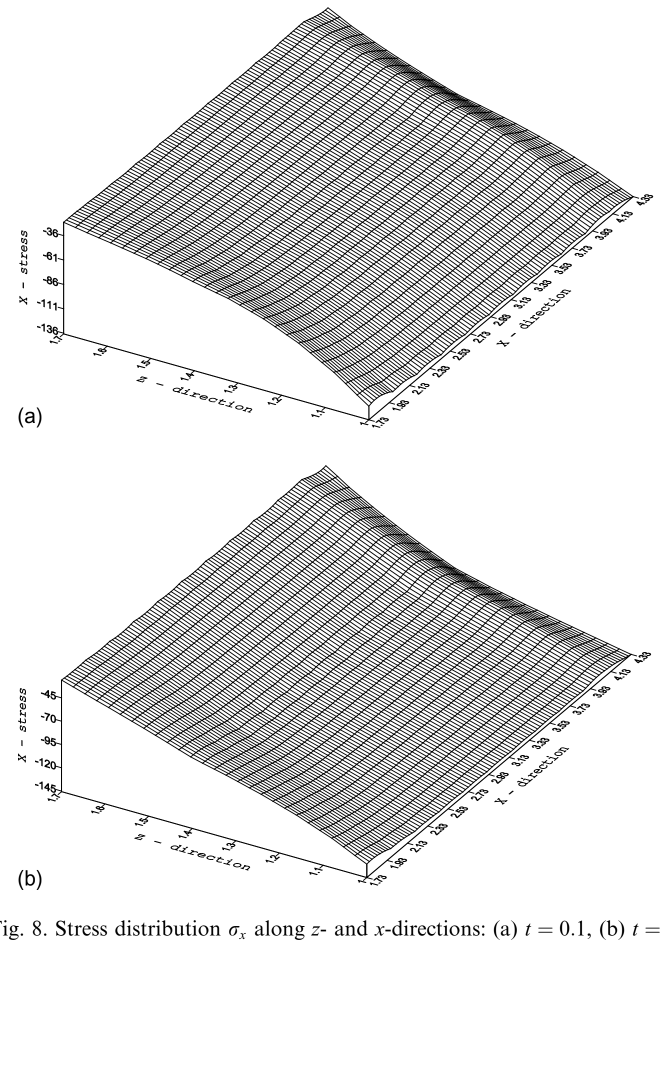

Fig. 8. Stress distribution σ_x along z- and x-directions: (a) $t = 0.1$, (b) $t = 5$.

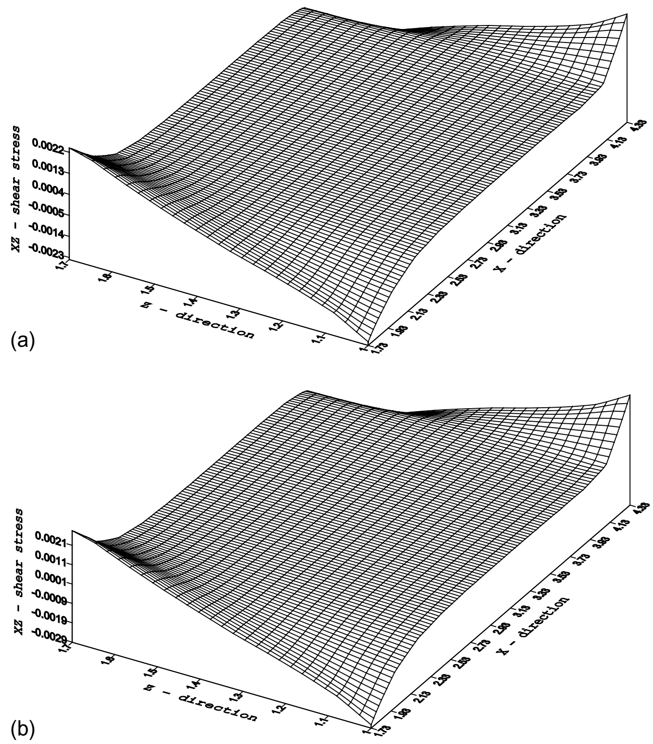

Fig. 9. Shear stress distribution τ_{xz} along z- and x-directions: (a) $t = 0.1$, (b) $t = 5$.

5. Conclusions

In this paper, the thermoelastic transient response of the laminated circular conical shell has been analyzed. The thermoelastic problem of circular conical shell composed of multilayer of different materials has also been discussed. The finite difference and the Laplace transform methods were employed to obtain the numerical results. Application of the present method to laminated circular conical shells reveals that the present method is rapidly convergent, we chose gird points $= 196$ ($N = 14$) to evaluate the temperature distribution, displacement and thermal stresses in a laminated circular conical shell for small time $t = 0.1$, which is in transient state, and large time $t = 5.0$, which approaches the steady state. Temperature, displacement and thermal stress distributions were obtained which can be applied to design useful structures or machines in engineering applications. There is no limit to the number of layers in such a circular conical shell. The discontinuity of circumferential stress at the interfaces was found. It was found that the temperature distribution, the displacement and the thermal stresses vary slightly as time intervals increase.

Appendix A

The matrices $[M_1]$ to $[M_{10}]$ are given as follows:

The matrix $[M_1]$

$$
\begin{array}{c|ccccccccccccccccccccccccc}
 & 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & \cdots & \cdots & \cdots & 3N & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & N^2 \\
\hline
1 & b & c &&&& a \\
2 & d & \ddots & \ddots &&&& \ddots \\
\vdots && \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&& \ddots & \ddots & c &&&& \ddots \\
N &&&& d & b &&&&& a \\
N+1 & c2 &&&&& b2 & d2 &&&& a2 \\
N+2 && c4 &&&& h4 & b4 & d4 &&&& a4 \\
\vdots &&& \ddots &&&& \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&&& c4 &&&& \ddots & \ddots & d4 &&&& a4 \\
2N &&&&& c3 &&&& d3 & b3 &&&&& a3 \\
\vdots &&&&&& c2 &&&&& b2 & d2 &&&& a2 \\
\vdots &&&&&&& c4 &&&& h4 & b4 & d4 &&&& a4 \\
\vdots &&&&&&&& \ddots &&&& \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&&&&&&&& c4 &&&& \ddots & \ddots & d4 &&&& a4 \\
3N &&&&&&&&&& c3 &&&& d3 & b3 &&&&& a3 \\
\vdots &&&&&&&&&&& c2 &&&&& b2 & d2 &&&& a2 \\
\vdots &&&&&&&&&&&& c4 &&&& h4 & b4 & d4 &&&& a4 \\
\vdots &&&&&&&&&&&&& \ddots &&&& \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&&&&&&&&&&&&& c4 &&&& \ddots & \ddots & d4 &&&& a4 \\
\vdots &&&&&&&&&&&&&&& c3 &&&& d3 & b3 &&&&& a3 \\
\vdots &&&&&&&&&&&&&&&& b1 &&&&& a1 & c1 \\
\vdots &&&&&&&&&&&&&&&&& \ddots &&&& d1 & \ddots & \ddots \\
\vdots &&&&&&&&&&&&&&&&&& \ddots &&&& \ddots & \ddots & \ddots \\
\vdots &&&&&&&&&&&&&&&&&&& \ddots &&&& \ddots & \ddots & c1 \\
N^2 &&&&&&&&&&&&&&&&&&&& b1 &&&& d1 & a1
\end{array}
$$

where

$$a = \left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\frac{a_{31}}{\Delta z^2} + \frac{a_{41}}{2\Delta z}\frac{1}{1 + x_{i,1} + z_{i,1}}\right], \quad b = -\left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\frac{2a_{11}}{\Delta x^2} + \frac{2a_{31}}{\Delta z^2}\right]$$

$$c = \left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\frac{a_{11}}{\Delta x^2} + \frac{a_{21}}{2\Delta x}\frac{1}{1 + x_{i,1} + z_{i,1}}\right], \quad d = \left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\frac{a_{11}}{\Delta x^2} - \frac{a_{21}}{2\Delta x}\frac{1}{1 + x_{i,1} + z_{i,1}}\right]$$

$$a2 = \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\frac{a_{3k}}{\Delta z^2} + \frac{a_{4k}}{2\Delta z}\frac{1}{1 + x_{1,j} + z_{1,j}}\right], \quad b2 = b3 = -\left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\frac{2a_{1k}}{\Delta x^2} + \frac{2a_{3k}}{\Delta z^2}\right]$$

$$c2 = \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\frac{a_{3k}}{\Delta z^2} - \frac{a_{4k}}{2\Delta z}\frac{1}{1 + x_{1,j} + z_{1,j}}\right], \quad d2 = \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\frac{a_{1k}}{\Delta x^2} + \frac{a_{2k}}{2\Delta x}\frac{1}{1 + x_{1,j} + z_{1,j}}\right]$$

$$a3 = \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\frac{a_{3k}}{\Delta z^2} + \frac{a_{4k}}{2\Delta z}\frac{1}{1 + x_{N,j} + z_{N,j}}\right], \quad c3 = \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\frac{a_{3k}}{\Delta z^2} - \frac{a_{4k}}{2\Delta z}\frac{1}{1 + x_{N,j} + z_{N,j}}\right]$$

$$d3 = \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\frac{a_{1k}}{\Delta x^2} - \frac{a_{2k}}{2\Delta x}\frac{1}{1 + x_{N,j} + z_{N,j}}\right], \quad a4 = \frac{a_{3k}}{\Delta z^2} + \frac{a_{4k}}{2\Delta z}\frac{1}{1 + x_{i,j} + z_{i,j}}$$

$$b4 = -\frac{2a_{1k}}{\Delta x^2} - \frac{2a_{3k}}{\Delta z^2}, \quad c4 = \frac{a_{3k}}{\Delta z^2} - \frac{a_{4k}}{2\Delta z}\frac{1}{1 + x_{i,j} + z_{i,j}}, \quad d4 = \frac{a_{1k}}{\Delta x^2} + \frac{a_{2k}}{2\Delta x}\frac{1}{1 + x_{i,j} + z_{i,j}}$$

$$h4 = \frac{a_{1k}}{\Delta x^2} - \frac{a_{2k}}{2\Delta x}\frac{1}{1 + x_{i,j} + z_{i,j}}, \quad a1 = -\left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\frac{2a_{1m}}{\Delta x^2} + \frac{2a_{3m}}{\Delta z^2}\right]$$

$$b1 = \left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\frac{a_{3m}}{\Delta z^2} - \frac{a_{4m}}{2\Delta z}\frac{1}{1 + x_{i,N} + z_{i,N}}\right]$$

$$c1 = \left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\frac{a_{1m}}{\Delta x^2} + \frac{a_{2m}}{2\Delta x}\frac{1}{1 + x_{i,N} + z_{i,N}}\right]$$

$$d1 = \left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\frac{a_{1m}}{\Delta x^2} - \frac{a_{2m}}{2\Delta x}\frac{1}{1 + x_{i,N} + z_{i,N}}\right]$$

where N is the mesh number; m is the last layer; $k = 2, 3, \ldots, m-1$ layer.

The matrix $[M_2]$

$$\begin{array}{c|ccccccccccccccccccccccccc}
 & 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & \cdots & \cdots & \cdots & 3N & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & N^2 \\
1 & e \\
2 & & \ddots \\
\vdots & & & \ddots \\
\vdots & & & & \ddots \\
N & & & & & e \\
N+1 & f2 & & & & & g2 & & & & & e2 \\
N+2 & & g4 & & & & & f4 & & & & & e4 \\
\vdots & & & \ddots & & & & & \ddots & & & & & \ddots \\
\vdots & & & & g4 & & & & & f4 & & & & & e4 \\
2N & & & & & g3 & & & & & f3 & & & & & e3 \\
\vdots & & & & & & f2 & & & & & g2 & & & & & e2 \\
\vdots & & & & & & & g4 & & & & & f4 & & & & & e4 \\
\vdots & & & & & & & & \ddots & & & & & \ddots & & & & & \ddots \\
\vdots & & & & & & & & & g4 & & & & & f4 & & & & & e4 \\
3N & & & & & & & & & & g3 & & & & & f3 & & & & & e3 \\
\vdots & & & & & & & & & & & f2 & & & & & g2 & & & & & e2 \\
\vdots & & & & & & & & & & & & g4 & & & & & f4 & & & & & e4 \\
\vdots & & & & & & & & & & & & & \ddots & & & & & \ddots & & & & & \ddots \\
\vdots & & & & & & & & & & & & & & g4 & & & & & f4 & & & & & e4 \\
\vdots & & & & & & & & & & & & & & & g3 & & & & & f3 & & & & & e3 \\
\vdots & & & & & & & & & & & & & & & & & & & & & e1 \\
\vdots & & & & & & & & & & & & & & & & & & & & & & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & & & & & & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & & & & & & & \ddots \\
N^2 & & & & & & & & & & & & & & & & & & & & & & & & & e1
\end{array}$$

where

$$e = \frac{-1}{1 + x_{i,1} + z_{i,1}} \left[1 + a_{71} \frac{{}_5P_1}{{}_4P_1}\right]^{-1} \left[a_{61} - a_{71} \frac{{}_3P_1}{{}_4P_1}\right], \quad e2 = e3 = -\left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1} \left[\frac{a_{7k}}{2\Delta z} - \frac{1}{2\Delta z} \frac{{}_4Q_k}{{}_1Q_k}\right]$$

$$f2 = g3 = -\left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1} \left[\frac{1}{2\Delta z} \frac{{}_4Q_k}{{}_1Q_k} - \frac{a_{7k}}{2\Delta z}\right], \quad g2 = \frac{-1}{1 + x_{1,j} + z_{1,j}} \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1} \left[a_{6k} - \frac{{}_3Q_k}{{}_1Q_k}\right]$$

$$f3 = \frac{-1}{1 + x_{N,j} + z_{N,j}} \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1} \left[a_{6k} - \frac{{}_3Q_k}{{}_1Q_k}\right], \quad e4 = \frac{-a_{7k}}{2\Delta z}, \quad f4 = \frac{-a_{6k}}{1 + x_{i,j} + z_{i,j}}, \quad g4 = \frac{a_{7k}}{2\Delta z}$$

$$e1 = \frac{-1}{1 + x_{i,N} + z_{i,N}} \left[1 + a_{7m} \frac{{}_5P_m}{{}_4P_m}\right]^{-1} \left[a_{6m} - a_{7m} \frac{{}_3P_m}{{}_4P_m}\right]$$

The matrix $[M_3]$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ \vdots \\ N \\ N+1 \\ N+2 \\ \vdots \\ \vdots \\ 2N \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 3N \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ N^2 \end{array}
\begin{array}{c} 1 \;\; 2 \;\cdots\;\cdots\; N \;\; N+1 \;\; N+2 \;\cdots\;\cdots\; 2N \;\cdots\;\cdots\;\cdots\;\cdots\; 3N \;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots\; N^2 \\
\left(\begin{array}{cccccccccccccccccccccc}
g & f \\
h & \ddots & \ddots \\
 & \ddots & \ddots & \ddots \\
 & & \ddots & \ddots & f \\
 & & & h & g \\
 & & & & & i2 \\
 & & & & & j4 & k4 & i4 \\
 & & & & & & \ddots & \ddots & \ddots \\
 & & & & & & & j4 & k4 & i4 \\
 & & & & & & & & & i3 \\
 & & & & & & & & & & i2 \\
 & & & & & & & & & & j4 & k4 & i4 \\
 & & & & & & & & & & & \ddots & \ddots & \ddots \\
 & & & & & & & & & & & & j4 & k4 & i4 \\
 & & & & & & & & & & & & & & i3 \\
 & & & & & & & & & & & & & & & i2 \\
 & & & & & & & & & & & & & & & j4 & k4 & i4 \\
 & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots \\
 & & & & & & & & & & & & & & & & & j4 & k4 & i4 \\
 & & & & & & & & & & & & & & & & & & & i3 \\
 & & & & & & & & & & & & & & & & & & & & g1 & f1 \\
 & & & & & & & & & & & & & & & & & & & & h1 & \ddots & \ddots \\
 & & & & & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots \\
 & & & & & & & & & & & & & & & & & & & & & & \ddots & \ddots & f1 \\
 & & & & & & & & & & & & & & & & & & & & & & & h1 & g1
\end{array}\right) \end{array}$$

where

$$f = -\left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\frac{1}{2\Delta x} - \frac{a_{71}}{2\Delta x}\frac{{}_1P_1}{{}_4P_1}\right], \quad g = \frac{-1}{1 + x_{i,1} + z_{i,1}}\left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[a_{51} - a_{71}\frac{{}_2P_1}{{}_4P_1}\right]$$

$$h = -\left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\frac{a_{71}}{2\Delta x}\frac{{}_1P_1}{{}_4P_1} - \frac{1}{2\Delta x}\right], \quad i2 = \frac{-1}{1 + x_{1,j} + z_{1,j}}\left[1 + \frac{{}_5Q_k}{{}_4Q_k}\right]^{-1}\left[a_{5k} - \frac{{}_2Q_k}{{}_1Q_k}\right]$$

$$i3 = \frac{-1}{1 + x_{N,j} + z_{N,j}}\left[1 + \frac{{}_5Q_k}{{}_4Q_k}\right]^{-1}\left[a_{5k} - \frac{{}_2Q_k}{{}_1Q_k}\right], \quad i4 = \frac{-1}{2\Delta x}, \quad j4 = \frac{1}{2\Delta x}$$

$$k4 = \frac{-a_{5k}}{1 + x_{i,j} + z_{i,j}}, \quad f1 = -\left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\frac{1}{2\Delta x} - \frac{a_{7m}}{2\Delta x}\frac{{}_1P_m}{{}_4P_m}\right]$$

$$g1 = \frac{-1}{1 + x_{i,N} + z_{i,N}}\left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[a_{5m} - a_{7m}\frac{{}_2P_m}{{}_4P_m}\right]$$

$$h1 = -\left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\frac{a_{7m}}{2\Delta x}\frac{{}_1P_m}{{}_4P_m} - \frac{1}{2\Delta x}\right]$$

The column matrix $[M_4]^{\mathrm{T}}$

1	2	···	···	N	$N+1$	$N+2$	···	···	$2N$	···	···	···	···	$3N$	···	···	···	···	···	···	···	···	···	N^2
[$v1$	$p1$	···	$p1$	$v2$	$z1$	0	···	0	$z2$	$z1$	0	···	0	$z2$	$z1$	0	···	0	$z2$	$v3$	$p2$	···	$p2$	$v4$]

where

$$p1 = -\left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\left(\frac{a_{31}}{\Delta z^2} - \frac{a_{41}}{2\Delta z}\frac{1}{1 + x_{i,1} + z_{i,1}}\right)\frac{f_1}{\Theta_0}\right]$$

$$v1 = \frac{-1}{\Theta_0}\left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\left(\frac{a_{11}}{\Delta x^2} - \frac{a_{21}}{2\Delta x}\frac{1}{1 + x_{1,1} + z_{1,1}}\right)f_4 + \left(\frac{a_{31}}{\Delta z^2} - \frac{a_{41}}{2\Delta z}\frac{1}{1 + x_{1,1} + z_{1,1}}\right)f_1\right]$$

$$v2 = \frac{-1}{\Theta_0}\left[1 + a_{71}\frac{{}_5P_1}{{}_4P_1}\right]^{-1}\left[\left(\frac{a_{11}}{\Delta x^2} + \frac{a_{21}}{2\Delta x}\frac{1}{1 + x_{N,1} + z_{N,1}}\right)f_3 + \left(\frac{a_{31}}{\Delta z^2} - \frac{a_{41}}{2\Delta z}\frac{1}{1 + x_{N,1} + z_{N,1}}\right)f_1\right]$$

$$z1 = \left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\left(\frac{a_{2k}}{2\Delta x}\frac{1}{1 + x_{1,j} + z_{1,j}} - \frac{a_{1k}}{\Delta x^2}\right)\frac{f_4}{\Theta_0}\right]$$

$$z2 = -\left[1 + \frac{{}_5Q_k}{{}_1Q_k}\right]^{-1}\left[\left(\frac{a_{1k}}{\Delta x^2} + \frac{a_{2k}}{2\Delta x}\frac{1}{1 + x_{N,j} + z_{N,j}}\right)\frac{f_3}{\Theta_0}\right]$$

$$p2 = -\left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\left(\frac{a_{3m}}{\Delta z^2} + \frac{a_{4m}}{2\Delta z}\frac{1}{1 + x_{i,N} + z_{i,N}}\right)\frac{f_2}{\Theta_0}\right]$$

$$v3 = \frac{-1}{\Theta_0}\left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\left(\frac{a_{1m}}{\Delta x^2} - \frac{a_{2m}}{2\Delta x}\frac{1}{1 + x_{1,N} + z_{1,N}}\right)f_4 + \left(\frac{a_{3m}}{\Delta z^2} + \frac{a_{4m}}{2\Delta z}\frac{1}{1 + x_{1,N} + z_{1,N}}\right)f_2\right]$$

$$v4 = \frac{-1}{\Theta_0}\left[1 + a_{7m}\frac{{}_5P_m}{{}_4P_m}\right]^{-1}\left[\left(\frac{a_{1m}}{\Delta x^2} + \frac{a_{2m}}{2\Delta x}\frac{1}{1 + x_{N,N} + z_{N,N}}\right)f_3 + \left(\frac{a_{3m}}{\Delta z^2} + \frac{a_{4m}}{2\Delta z}\frac{1}{1 + x_{N,N} + z_{N,N}}\right)f_2\right]$$

The matrix $[M_5]$

$$
\begin{array}{c|ccccccccccccccccccccccccc}
 & 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & \cdots & \cdots & \cdots & 3N & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & N^2 \\
\hline
1 & b5 & a5 \\
2 & c5 & \ddots & a5 \\
\vdots & & \ddots & \ddots & \ddots \\
\vdots & & & \ddots & \ddots & a5 \\
N & & & & c5 & \ddots \\
N+1 & & & & & & \ddots & a5 \\
N+2 & & & & & & c5 & \ddots & a5 \\
\vdots & & & & & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & \ddots & \ddots & a5 \\
2N & & & & & & & & & c5 & \ddots \\
\vdots & & & & & & & & & & & \ddots & a5 \\
\vdots & & & & & & & & & & & c5 & \ddots & a5 \\
\vdots & & & & & & & & & & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & \ddots & \ddots & a5 \\
3N & & & & & & & & & & & & & & c5 & \ddots \\
\vdots & & & & & & & & & & & & & & & & \ddots & a5 \\
\vdots & & & & & & & & & & & & & & & & c5 & \ddots & a5 \\
\vdots & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & \ddots & \ddots & a5 \\
\vdots & & & & & & & & & & & & & & & & & & & c5 & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & & & & \ddots & a5 \\
\vdots & & & & & & & & & & & & & & & & & & & & & c5 & \ddots & a5 \\
\vdots & & & & & & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & & & & & & \ddots & \ddots & a5 \\
N^2 & & & & & & & & & & & & & & & & & & & & & & & & c5 & b5
\end{array}
$$

where

$$a5 = \frac{-b_{8k}}{2\Delta x}, \quad b5 = \frac{-b_{9k}}{1 + x_{i,j} + z_{i,j}}, \quad c5 = \frac{b_{8k}}{2\Delta x}$$

The matrix $[M_6]$

$$
\begin{array}{c|ccccccccccccccccccccccccc}
 & 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & \cdots & \cdots & \cdots & 3N & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & N^2 \\
1 & f5 & h5 & & & & e5 & i5 \\
2 & j5 & \ddots & \ddots & & & k5 & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & \ddots & \ddots & h5 & & & \ddots & \ddots & i5 \\
N & & & & j5 & \ddots & & & & k5 & \ddots \\
N+1 & g5 & m5 & & & & \ddots & h5 & & & & \ddots & i5 \\
N+2 & n5 & \ddots & \ddots & & & j5 & \ddots & \ddots & & & k5 & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & \ddots & \ddots & m5 & & & \ddots & \ddots & h5 & & & \ddots & \ddots & i5 \\
2N & & & & n5 & \ddots & & & & j5 & \ddots & & & & k5 & \ddots \\
\vdots & & & & & & \ddots & m5 & & & & \ddots & h5 & & & & \ddots & i5 \\
\vdots & & & & & & n5 & \ddots & \ddots & & & j5 & \ddots & \ddots & & & k5 & \ddots & \ddots \\
\vdots & & & & & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & \ddots & \ddots & m5 & & & \ddots & \ddots & h5 & & & \ddots & \ddots & i5 \\
3N & & & & & & & & & n5 & \ddots & & & & j5 & \ddots & & & & k5 & \ddots \\
\vdots & & & & & & & & & & & \ddots & m5 & & & & \ddots & h5 & & & & \ddots & i5 \\
\vdots & & & & & & & & & & & n5 & \ddots & \ddots & & & j5 & \ddots & \ddots & & & k5 & \ddots & \ddots \\
\vdots & & & & & & & & & & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & \ddots & \ddots & m5 & & & \ddots & \ddots & h5 & & & \ddots & \ddots & i5 \\
\vdots & & & & & & & & & & & & & & n5 & \ddots & & & & j5 & \ddots & & & & k5 & e5 \\
\vdots & & & & & & & & & & & & & & & & \ddots & m5 & & & & \ddots & h5 \\
\vdots & & & & & & & & & & & & & & & & n5 & \ddots & \ddots & & & j5 & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & \ddots & \ddots & m5 & & & \ddots & \ddots & h5 \\
N^2 & & & & & & & & & & & & & & & & & & & n5 & g5 & & & & j5 & f5
\end{array}
$$

where

$$e5 = \frac{1}{1 + x_{i,j} + z_{i,j}} \frac{b_{4k}}{2\Delta z}, \quad f5 = \frac{-b_{5k}}{(1 + x_{i,j} + z_{i,j})^2}, \quad g5 = \frac{-1}{1 + x_{i,j} + z_{i,j}} \frac{b_{4k}}{2\Delta z}$$

$$h5 = \frac{1}{1 + x_{i,j} + z_{i,j}} \frac{b_{6k}}{2\Delta x}, \quad i5 = n5 = \frac{b_{7k}}{4\Delta x \Delta z}, \quad j5 = \frac{-1}{1 + x_{i,j} + z_{i,j}} \frac{b_{6k}}{2\Delta x}, \quad k5 = m5 = \frac{-b_{7_k}}{4\Delta x \Delta z}$$

The matrix $[M_7]$

$$
\begin{array}{c|ccccccccccccccccccccccccc}
 & 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & \cdots & \cdots & \cdots & 3N & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & N^2 \\
1 & p5 & q5 &&&& r5 \\
2 & s5 & \ddots & \ddots &&&& \ddots \\
\vdots && \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&& \ddots & \ddots & q5 &&&& \ddots \\
N &&&& s5 & \ddots &&&&& \ddots \\
N+1 & u5 &&&&& \ddots & q5 &&&& \ddots \\
N+2 && \ddots &&&& s5 & \ddots & \ddots &&&& \ddots \\
\vdots &&& \ddots &&&& \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&&& \ddots &&&& \ddots & \ddots & q5 &&&& \ddots \\
2N &&&&& \ddots &&&& s5 & \ddots &&&&& \ddots \\
\vdots &&&&&& \ddots &&&&& \ddots & q5 &&&& \ddots \\
\vdots &&&&&&& \ddots &&&& s5 & \ddots & \ddots &&&& \ddots \\
\vdots &&&&&&&& \ddots &&&& \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&&&&&&&& \ddots &&&& \ddots & \ddots & q5 &&&& \ddots \\
3N &&&&&&&&&& \ddots &&&& s5 & \ddots &&&&& \ddots \\
\vdots &&&&&&&&&&& \ddots &&&&& \ddots & q5 &&&& \ddots \\
\vdots &&&&&&&&&&&& \ddots &&&& s5 & \ddots & \ddots &&&& \ddots \\
\vdots &&&&&&&&&&&&& \ddots &&&& \ddots & \ddots & \ddots &&&& \ddots \\
\vdots &&&&&&&&&&&&&& \ddots &&&& \ddots & \ddots & q5 &&&& \ddots \\
\vdots &&&&&&&&&&&&&&& \ddots &&&& s5 & \ddots &&&&& r5 \\
\vdots &&&&&&&&&&&&&&&& \ddots &&&&& \ddots & q5 \\
\vdots &&&&&&&&&&&&&&&&& \ddots &&&& s5 & \ddots & \ddots \\
\vdots &&&&&&&&&&&&&&&&&& \ddots &&&& \ddots & \ddots & \ddots \\
\vdots &&&&&&&&&&&&&&&&&&& \ddots &&&& \ddots & \ddots & q5 \\
N^2 &&&&&&&&&&&&&&&&&&&& u5 &&&& s5 & b5
\end{array}
$$

where

$$
p5 = \frac{-2}{\Delta x^2} - \frac{2b_{2k}}{\Delta z^2} - \frac{b_{1k}}{(1 + x_{i,j} + z_{i,j})^2}, \quad q5 = \frac{1}{2\Delta x}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{1}{\Delta x^2}
$$

$$
r5 = \frac{b_{3k}}{2\Delta z}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{b_{2k}}{\Delta z^2}, \quad s5 = \frac{-1}{2\Delta x}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{1}{\Delta x^2}
$$

$$u5 = \frac{-b_{3k}}{2\Delta z}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{b_{2k}}{\Delta z^2}$$

The matrix $[M_8]$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ \vdots \\ N \\ N+1 \\ N+2 \\ \vdots \\ \vdots \\ 2N \\ \vdots \\ \vdots \\ 3N \\ \vdots \\ \vdots \\ \vdots \\ N^2 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & 3N & \cdots & N^2 \end{array} \\
\begin{pmatrix}
b6 & & & & & a6 & & & & & & & & \\
 & \ddots & & & & & \ddots & & & & & & & \\
 & & \ddots & & & & & \ddots & & & & & & \\
 & & & \ddots & & & & & \ddots & & & & & \\
c6 & & & & \ddots & & & & & \ddots & & & & \\
 & \ddots & & & & \ddots & & & & & \ddots & & & \\
 & & \ddots & & & & \ddots & & & & & \ddots & & \\
 & & & \ddots & & & & \ddots & & & & & \ddots & \\
 & & & & \ddots & & & & \ddots & & & & & a6 \\
 & & & & & \ddots & & & & \ddots & & & & \\
 & & & & & & \ddots & & & & \ddots & & & \\
 & & & & & & & \ddots & & & & \ddots & & \\
 & & & & & & & & c6 & & & & & b6
\end{pmatrix}
\end{array}$$

where

$$a6 = -\frac{c_{8k}}{2\Delta z}, \quad b6 = \frac{-c_{9k}}{1 + x_{i,j} + z_{i,j}}, \quad c6 = \frac{c_{8k}}{2\Delta z}$$

The matrix $[M_9]$

$$
\begin{array}{c|ccccccccccccccccccccccccc}
 & 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & \cdots & \cdots & \cdots & 3N & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & N^2 \\
\hline
1 & f6 & h6 & & & & e6 \\
2 & j6 & \ddots & \ddots & & & & \ddots \\
\vdots & & \ddots & \ddots & \ddots & & & & \ddots \\
\vdots & & & \ddots & \ddots & h6 & & & & \ddots \\
N & & & & j6 & \ddots & & & & & \ddots \\
N+1 & g6 & & & & & \ddots & h6 & & & & \ddots \\
N+2 & & \ddots & & & & j6 & \ddots & \ddots & & & & \ddots \\
\vdots & & & \ddots & & & & \ddots & \ddots & \ddots & & & & \ddots \\
\vdots & & & & \ddots & & & & \ddots & \ddots & h6 & & & & \ddots \\
2N & & & & & \ddots & & & & j6 & \ddots & & & & & \ddots \\
\vdots & & & & & & \ddots & & & & & \ddots & h6 & & & & \ddots \\
\vdots & & & & & & & \ddots & & & & j6 & \ddots & \ddots & & & & \ddots \\
\vdots & & & & & & & & \ddots & & & & \ddots & \ddots & \ddots & & & & \ddots \\
\vdots & & & & & & & & & \ddots & & & & \ddots & \ddots & h6 & & & & \ddots \\
3N & & & & & & & & & & \ddots & & & & j6 & \ddots & & & & & \ddots \\
\vdots & & & & & & & & & & & \ddots & & & & & \ddots & h6 & & & & \ddots \\
\vdots & & & & & & & & & & & & \ddots & & & & j6 & \ddots & \ddots & & & & \ddots \\
\vdots & & & & & & & & & & & & & \ddots & & & & \ddots & \ddots & \ddots & & & & \ddots \\
\vdots & & & & & & & & & & & & & & \ddots & & & & \ddots & \ddots & h6 & & & & \ddots \\
\vdots & & & & & & & & & & & & & & & \ddots & & & & j6 & \ddots & & & & & e6 \\
\vdots & & & & & & & & & & & & & & & & \ddots & & & & & \ddots & h6 \\
\vdots & & & & & & & & & & & & & & & & & \ddots & & & & j6 & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & \ddots & & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & & \ddots & & & & \ddots & \ddots & h6 \\
N^2 & & & & & & & & & & & & & & & & & & & & g6 & & & & j6 & f6
\end{array}
$$

where

$$e6 = \frac{c_{3k}}{2\Delta z}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{c_{2k}}{\Delta z^2}, \quad f6 = \frac{-2}{\Delta x^2} - \frac{2c_{2k}}{\Delta z^2} - \frac{c_{1k}}{(1 + x_{i,j} + z_{i,j})^2}$$

$$g6 = \frac{-c_{3k}}{2\Delta z}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{c_{2k}}{\Delta z^2}, \quad h6 = \frac{1}{2\Delta x}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{1}{\Delta x^2}$$

$$j6 = \frac{-1}{2\Delta x}\frac{1}{1 + x_{i,j} + z_{i,j}} + \frac{1}{\Delta x^2}$$

The matrix $[M_{10}]$

$$
\begin{array}{rccccccccccccccccccccccccc}
 & 1 & 2 & \cdots & \cdots & N & N+1 & N+2 & \cdots & \cdots & 2N & \cdots & \cdots & \cdots & \cdots & 3N & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & N^2 \\
1 & p6 & q6 & & & & r6 & i6 \\
2 & s6 & \ddots & \ddots & & & k6 & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & \ddots & \ddots & q6 & & & \ddots & \ddots & i6 \\
N & & & & s6 & \ddots & & & & k6 & \ddots \\
N+1 & u6 & m6 & & & & \ddots & q6 & & & & \ddots & i6 \\
N+2 & n6 & \ddots & \ddots & & & s6 & \ddots & \ddots & & & k6 & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & \ddots & \ddots & m6 & & & \ddots & \ddots & q6 & & & \ddots & \ddots & i6 \\
2N & & & & n6 & \ddots & & & & s6 & \ddots & & & & k6 & \ddots \\
\vdots & & & & & & \ddots & m6 & & & & \ddots & q6 & & & & \ddots & i6 \\
\vdots & & & & & & n6 & \ddots & \ddots & & & s6 & \ddots & \ddots & & & k6 & \ddots & \ddots \\
\vdots & & & & & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & \ddots & \ddots & m6 & & & \ddots & \ddots & q6 & & & \ddots & \ddots & i6 \\
3N & & & & & & & & & n6 & \ddots & & & & s6 & \ddots & & & & k6 & \ddots \\
\vdots & & & & & & & & & & & \ddots & m6 & & & & \ddots & q6 & & & & \ddots & i6 \\
\vdots & & & & & & & & & & & n6 & \ddots & \ddots & & & s6 & \ddots & \ddots & & & k6 & \ddots & \ddots \\
\vdots & & & & & & & & & & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & \ddots & \ddots & m6 & & & \ddots & \ddots & q6 & & & \ddots & \ddots & i6 \\
\vdots & & & & & & & & & & & & & & n6 & \ddots & & & & s6 & \ddots & & & & k6 & r6 \\
\vdots & & & & & & & & & & & & & & & & \ddots & m6 & & & & \ddots & q6 \\
\vdots & & & & & & & & & & & & & & & & n6 & \ddots & \ddots & & & s6 & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & & & & & & & & & & & & & & & & & & \ddots & \ddots & m6 & & & \ddots & \ddots & q6 \\
N^2 & & & & & & & & & & & & & & & & & & & n6 & u6 & & & & s6 & p6
\end{array}
$$

where

$$i6 = n6 = \frac{c_{7k}}{4\Delta x\Delta z}, \quad k6 = m6 = \frac{-c_{7k}}{4\Delta x\Delta z}, \quad p6 = \frac{-c_{5k}}{(1+x_{i,j}+z_{i,j})^2}, \quad q6 = \frac{1}{1+x_{i,j}+z_{i,j}}\frac{c_{6k}}{2\Delta x}$$

$$r6 = \frac{1}{1+x_{i,j}+z_{i,j}}\frac{c_{4k}}{2\Delta z}, \quad s6 = \frac{-1}{1+x_{i,j}+z_{i,j}}\frac{c_{6k}}{2\Delta x}, \quad u6 = \frac{-1}{1+x_{i,j}+z_{i,j}}\frac{c_{4k}}{2\Delta z}$$

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